LIÈGE université
Sciences Appliquées

Lecture 4 The Diffusion Equation

Mathématiques appliquées (MATH0504-1)
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The Diffusion Equation

In this lecture we begin the study of the diffusion equation

$$u_t = ku_{xx}$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the PDEs.

Since solving the diffusion equation is harder than solving the wave equation, we start with a study of general properties of diffusions.

Learning objectives of this lecture

Learn general properties of the diffusion equation

Understand and prove the “maximum principle”

Prove the uniqueness and the stability of a diffusion problem

Outline

Maximum principle

Uniqueness for the Dirichlet problem for the diffusion equation

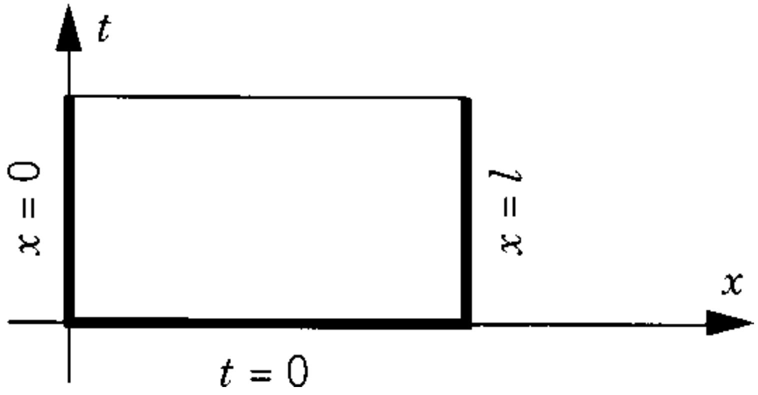
Stability of the Dirichlet problem

1 – Maximum Principle

Maximum Principle

If $u(x, t)$ satisfies the diffusion equation $u_t = k\, u_{xx}$ in a rectangle R ($0 \le x \le l,\ 0 \le t \le T$) in space-time, then the maximum value of $u(x, t)$ is assumed

- either initially ($t = 0$)
- or on the lateral sides ($x = 0$ or $x = l$)



Maximum Principle: interpretation

If you have a rod with no internal heat source, the hottest spot and the coldest spot can occur only initially or at one of the two ends of the rod

- Thus a hot spot at time zero will cool off (unless heat is fed into the rod at an end)
- You can burn one of its ends but the maximum temperature will always be at the hot end

Similarly, if you have a substance diffusing along a tube, its highest concentration can occur only initially or at one of the ends of the tube

Maximum Principle: interpretation

If we draw a “movie” of the solution, the maximum drops down while the minimum comes up.

So the diffusion equation tends to *smooth* the solution out.

This is very different from the behavior of the wave equation!

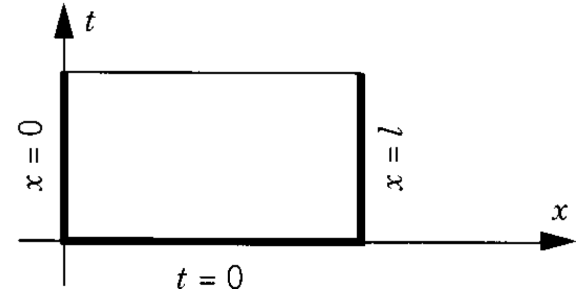
Maximum Principle: idea for a proof

If the maximum is at an interior point, we know that at this point we have $u_t = 0$ and $u_{xx} \leq 0$

First idea:

- If we could assume that $u_{xx} \neq 0$ at the maximum, then we would have $u_t \neq k u_{xx}$, which would contradict the diffusion equation; which would mean that the maximum must be on the boundary
- Unfortunately we *can* have $u_{xx} = 0$ at the maximum… So we need a slightly more elaborate strategy to prove the maximum principle

Maximum Principle: proof



Let M denote the maximum value of $u(x, t)$ on the three sides $t = 0$, $x = 0$, and $x = l$.

We must show that $u(x, t) \leq M$ throughout the rectangle R.

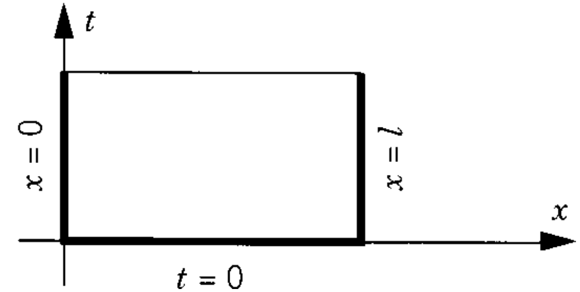
Let ϵ be a positive constant ($\epsilon > 0$) and let

$$v(x, t) = u(x, t) + \epsilon x^2$$

Our goal is to show that $v(x, t) \leq M + \epsilon l^2$ throughout R.

(Indeed, then $u(x, t) \leq M + \epsilon(l^2 - x^2)$ for any positive ϵ, which proves $u(x, t) \leq M$ throughout R)

Maximum Principle: proof



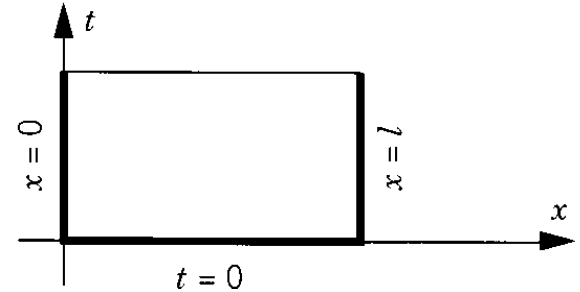
It is clear that $v(x,t) \leq M + \epsilon l^2$ on $t = 0$, $x = 0$ and $x = l$.

What about in the interior and on the top side?

The function $v(x,t) = u(x,t) + \epsilon x^2$ satisfies the "diffusion inequality"

$$\begin{aligned} v_t - k v_{xx} &= u_t - k(u + \epsilon x^2)_{xx} \\ &= u_t - k u_{xx} - 2\epsilon k \\ &= -2\epsilon k \\ &< 0 \end{aligned}$$

Maximum Principle: proof



Let's suppose that $v(x,t)$ attains a maximum at an interior point (x_0, t_0). Then $v_t = 0$ and $v_{xx} \leq 0$ at the point (x_0, t_0) : this violates the diffusion inequality.

Let's suppose that $v(x,t)$ attains a maximum on the top edge. Then

$$v_t(x_0, t_0) = \lim \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \geq 0$$

and $v_{xx} \leq 0$: this again violates the diffusion inequality.

Hence the maximum has to be on the bottom or the sides.

Minimum Principle

The minimum value has the same property: it too can be attained only on the bottom or the lateral sides.

To prove the minimum principle, just apply the maximum principle to $-u(x, t)$!

2 – Uniqueness

Uniqueness

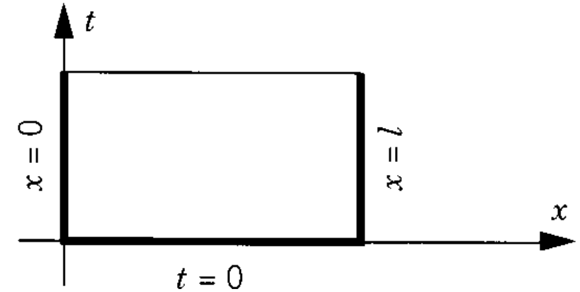
The maximum principle can be used to give a proof of *uniqueness for the Dirichlet problem for the diffusion equation.*

That is, there is at most one solution of

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad \text{for } 0 < x < l \text{ and } t > 0 \\ u(x,0) &= \phi(x) \\ u(0,t) &= g(t) \qquad u(l,t) = h(t) \end{aligned}$$

for the given functions f, ϕ, g and h

Uniqueness: proof



let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of the Dirichlet problem.

Let $w = u_1 - u_2$ be their difference.

Then $w_t - kw_{xx} = 0$, $w(x, 0) = w(0, t) = w(l, t) = 0$.

By the maximum principle, $w(x, t)$ has its maximum for the rectangle on its bottom or sides—exactly where it vanishes. So $w(x, t) \leq 0$.

The same type of argument for the minimum shows that $w(x, t) \geq 0$. Therefore, $w(x, t) \equiv 0$, so that $u_1(x, t) \equiv u_2(x, t)$ for all $t \geq 0$.

Uniqueness: alternative proof ("energy method")

Multiplying the equation for $w = u_1 - u_2$ by w itself, we can write:

$$0 = 0 \cdot w = (w_t - kw_{xx})(w) = \left(\tfrac{1}{2}w^2\right)_t + (-kw_x w)_x + kw_x^2$$

Integrating over $0 < x < l$, we get

$$0 = \int_0^l \left(\tfrac{1}{2}w^2\right)_t dx - kw_x w\Big|_{x=0}^{x=l} + k\int_0^l w_x^2\, dx$$

Since $w = 0$ at $x = 0$ and at $x = l$, we get

$$\frac{d}{dt}\int_0^l \frac{1}{2}[w(x,\ t)]^2\, dx = -k\int_0^l [w_x(x,\ t)]^2\, dx \le 0$$

Uniqueness: alternative proof ("energy method")

This means that $\int w^2\,dx$ is decreasing time, and thus:

$$\int_0^l [w(x,\ t)]^2\,dx \le \int_0^l [w(x,\ 0)]^2\,dx \quad \text{for } t \ge 0.$$

The right-hand side vanishes since the initial conditions on u and v are identical, i.e. $w(x, 0) = u(x, 0) - v(x, 0) = 0$.

Hence $\int [w(x,t)]^2\,dx = 0$ for $t > 0$, and thus

$w(x, t) \equiv 0$, so that $u_1(x, t) \equiv u_2(x, t)$ for all $t \geq 0$.

3 – Stability

Stability: using the maximum principle

Consider two solutions u_1 and u_2 of the Dirichlet diffusion problem with initial data ϕ_1 and ϕ_2 .

We have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle R and $w = \phi_1 - \phi_2$ on the bottom.

The maximum principle asserts that throughout the rectangle

$$u_1(x,t) - u_2(x,t) \leq \max|\phi_1 - \phi_2|$$

The “minimum” principle says that

$$u_1(x,t) - u_2(x,t) \geq -\max|\phi_1 - \phi_2|$$

Stability: using the maximum principle

Therefore:

$$\max_{0<x<l} |u_1(x,t) - u_2(x,t)| \le \max_{0<x<l} |\phi_1(x) - \phi_2(x)|$$

valid for all $t > 0$.

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at $t = 0$), we stay nearby.

This is a measure of stability; called stability in the "uniform" sense.

Stability: using the energy method

Consider again two solutions u_1 and u_2 of the Dirichlet problem with initial data ϕ_1 and ϕ_2, and with $f = g = h = 0$.

Then $w \equiv u_1 - u_2$ is the solution with initial data ϕ_1 - ϕ_2 and the inequality from slide 17

$$\int_0^l [w(x, t)]^2 \, dx \leq \int_0^l [w(x, 0)]^2 \, dx$$

becomes

$$\int_0^l [u_1(x, t) - u_2(x, t)]^2 \, dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2 \, dx$$

Stability: using the energy method

This again measures on the right the nearness of the initial data for two solutions, and on the left the nearness of the solutions at any later time.

This is another measure of stability; called stability in the "square integral" (or energy) sense.

Take-home messages

The maximum (and minimum) principle provide deep insights into the qualitative behavior of the solution of the diffusion equation

Diffusion tends to smooth the solution out

We can prove **uniqueness** and **stability** using this maximum principle

We can also prove uniqueness and stability using an "energy" method

What's next?

We still haven't *solved* the diffusion equation. We will do it in a future lecture, by deriving a formula for the solution on the whole line – which will introduce the concept of fundamental solutions (or *Green's functions*).

Before doing so, we will investigate *approximate solutions* using numerical methods.