Electromagnetic Energy Conversion ELEC0431

Exercise session 1: Phasors and power in the sinusoidal steady state

9 February 2024

Florent Purnode (florent.purnode@uliege.be)

Montefiore Institute, Department of Electrical Engineering and Computer Science,
University of Liège, Belgium

In this class...

Today's aim: Introduce a **very important** concept: The phasors

(pay particular attention to this class!)

- Reminders (one-ports and Kirchhoff's laws)
- Harmonic functions
- Root Mean Square value
- The phasors
- Exercise 1
- Product of two harmonic functions
- Active, reactive and apparent power
- Exercises 2

Reminders

One-ports

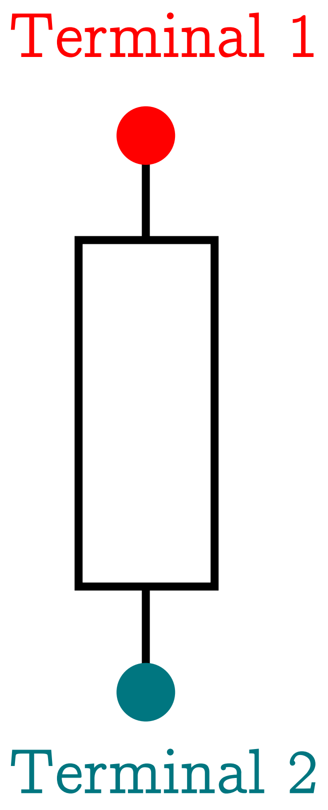
Kirchhoff laws

One-ports (or “dipoles” in French)

A one-port is a two-terminal electric component.

It can be:

- A resistance R
- An inductance L
- A capacitance C
- A power generator
- A diode
- A combination of components
- Etc.



One-ports (or "dipoles" in French)

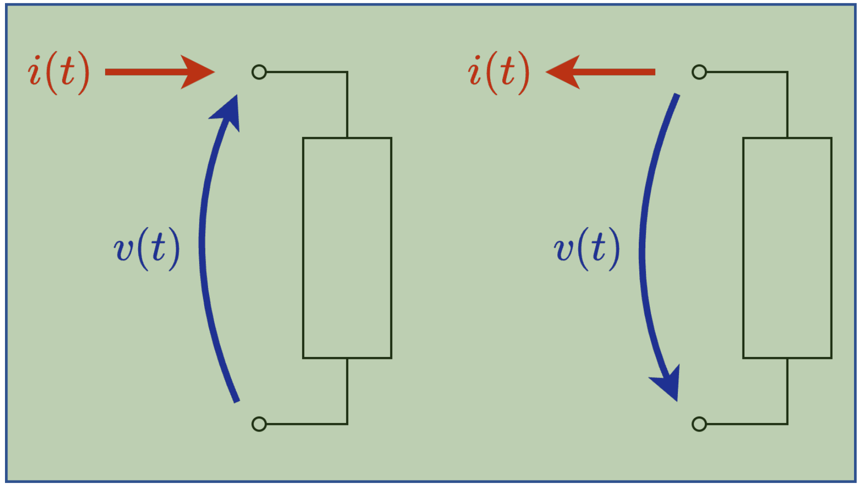
Each one-port is associated with a current $i(t)$ and a voltage $v(t)$.

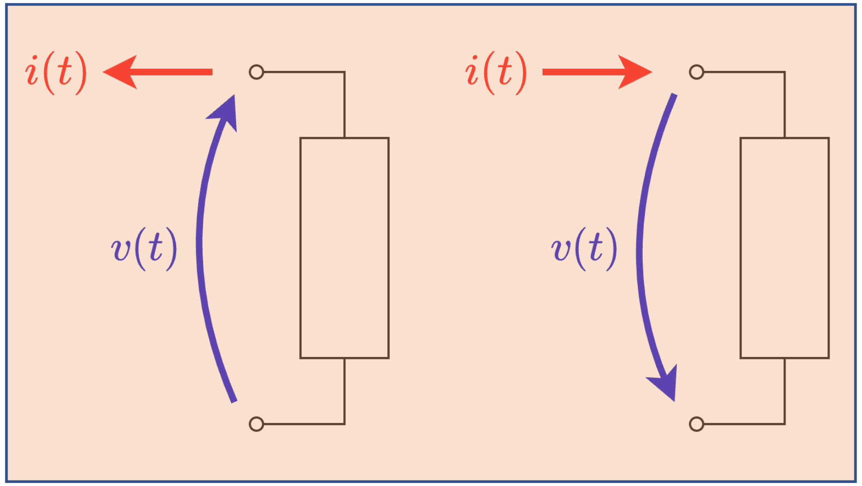
- ➢ $i(t)$ is positive if the positive charges flow in the direction indicated by the arrows.
- ➢ $v(t)$ is positive if the tension at the head of the arrow is larger than the one at the tip.

Four different configurations can thus be obtained.

➔ Two of them consume power and two of them deliver power.

Consume Power (like a **passive** component)





Deliver Power (like an **active** component)

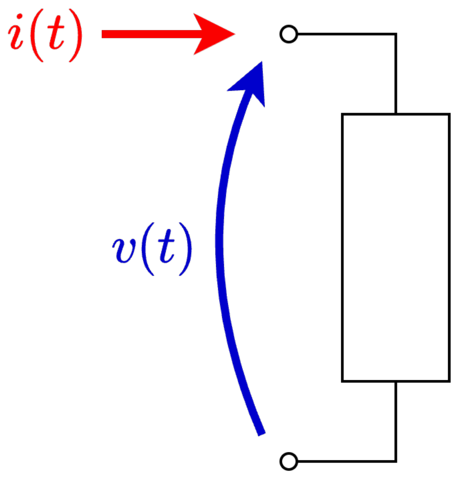
➔ This leads to two different conventions: the passive convention and the active convention.

Passive VS Active conventions

The passive convention

(also called receiver, motor or load convention)

The <u>passive</u> one-port **receives** the power $p(t) = v(t)\, i(t)$

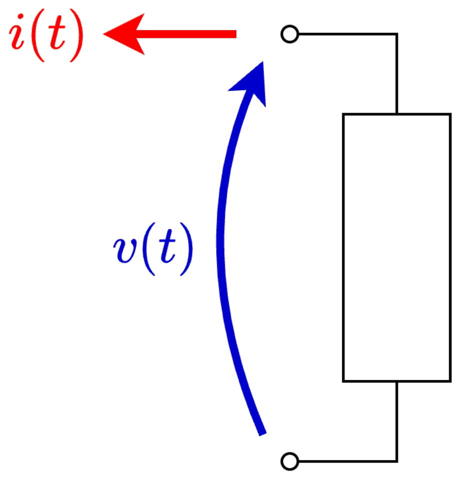


This convention is used for R, L, C, lumped parameters, motors and <u>primary</u> side of transformers.

The active convention

(also called generator or source convention)

The <u>active</u> one-port **delivers** the power $p(t) = v(t)\, i(t)$

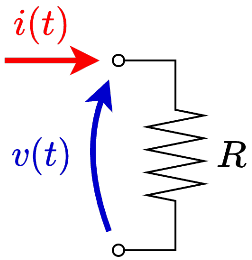


This convention is used for voltage sources, current sources, generator and <u>secondary</u> side of transformers.

$v(t)$ – i(t) relationship of basic components

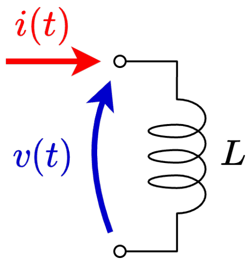
Passive convention

R



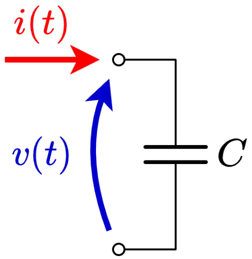
$$v(t) = R\,i(t)$$

L



$$v(t) = L\frac{di(t)}{dt}$$

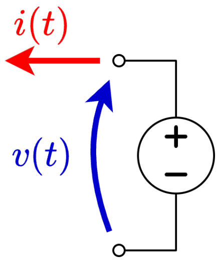
C



$$i(t) = C\frac{dv(t)}{dt}$$

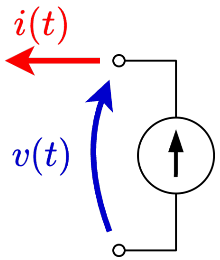
Active convention

Voltage source



$v(t)$ is imposed, $i(t)$ is set by the rest of the circuit

Current source



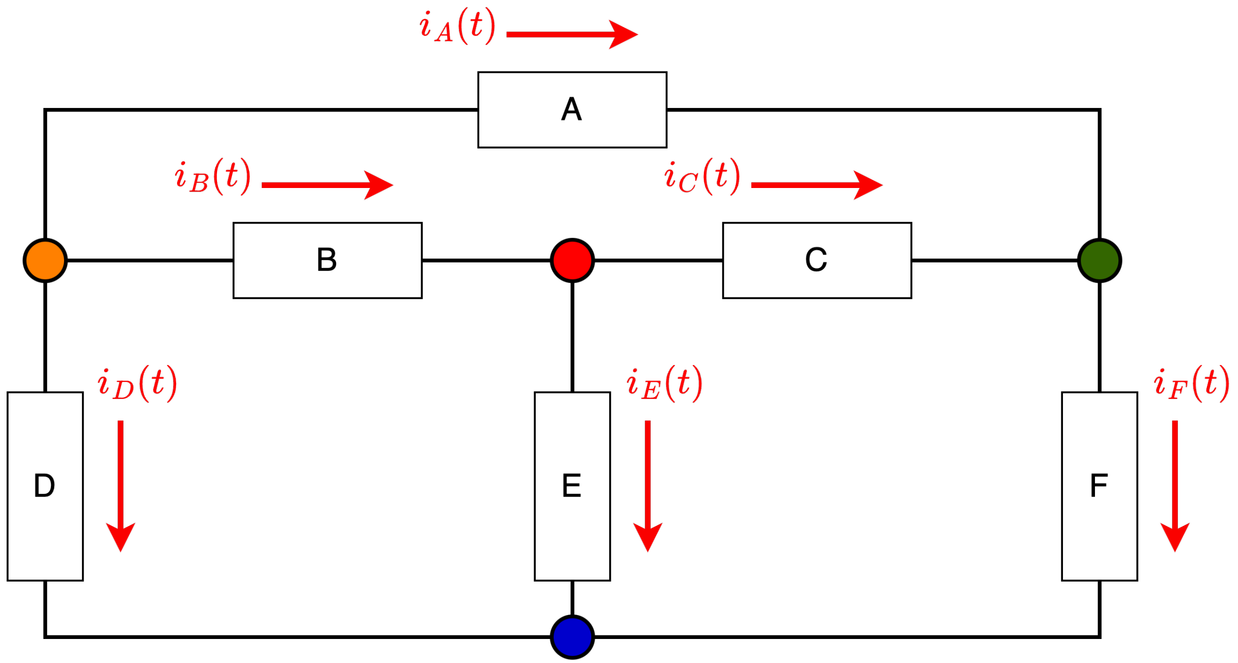
$i(t)$ is imposed, $v(t)$ is set by the rest of the circuit

When connected together, use Kirchhoff laws

Kirchhoff's first law

(also called Kirchhoff's junction rule or Kirchhoff's current law)

At any junction in the electrical circuit, the sum of currents flowing into the junction is equal to the sum of currents flowing out of the junction.



$$0 = i_A(t) + i_B(t) + i_D(t)$$

$$i_B(t) = i_C(t) + i_E(t)$$

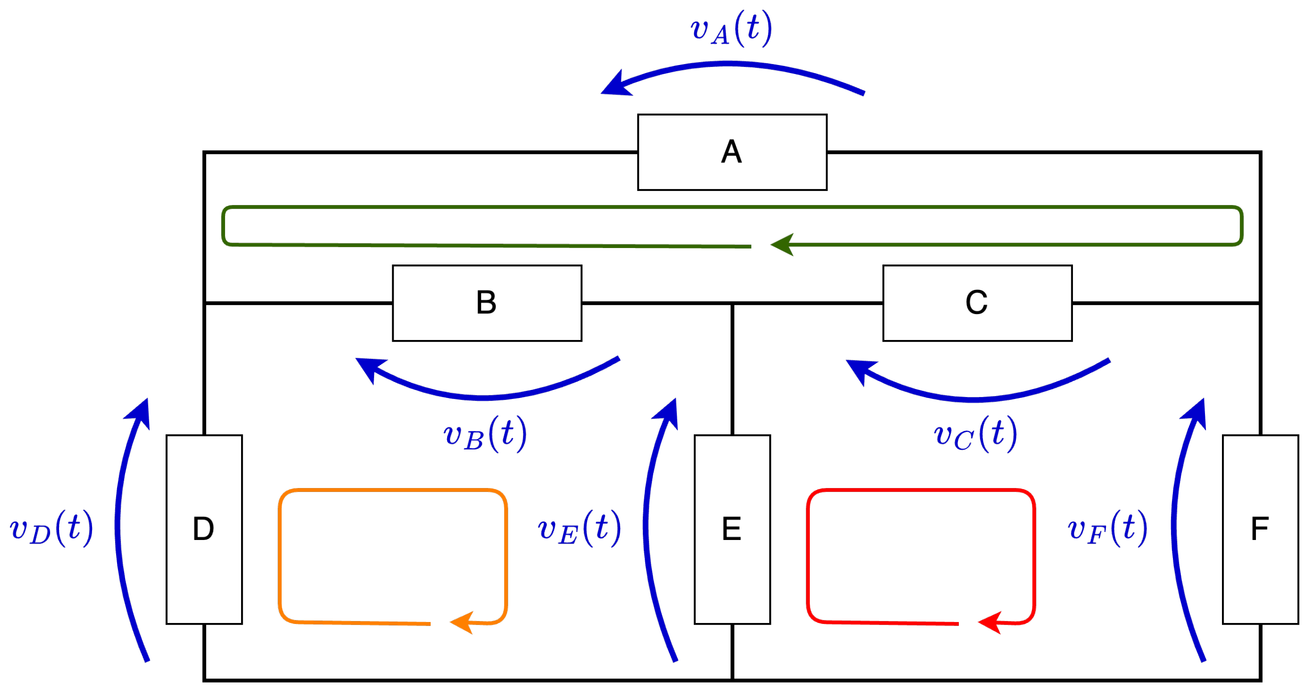
$$i_A(t) + i_C(t) = i_F(t)$$

$$i_D(t) + i_E(t) + i_F(t) = 0$$

Kirchhoff's second law

(also called Kirchhoff's loop rule or Kirchhoff's voltage law)

Around any closed loop in a circuit, the directed sum of potential differences across components is zero.



$$v_D(t) - v_B(t) - v_E(t) = 0$$

$$v_E(t) - v_C(t) - v_F(t) = 0$$

$$v_B(t) - v_A(t) + v_C(t) = 0$$

Introduction to phasors

Harmonic functions

Root Mean square value

The phasors

Harmonic function $\Leftrightarrow$ Phasor equivalence

Phasors applied to circuits

Application: Exercise 1

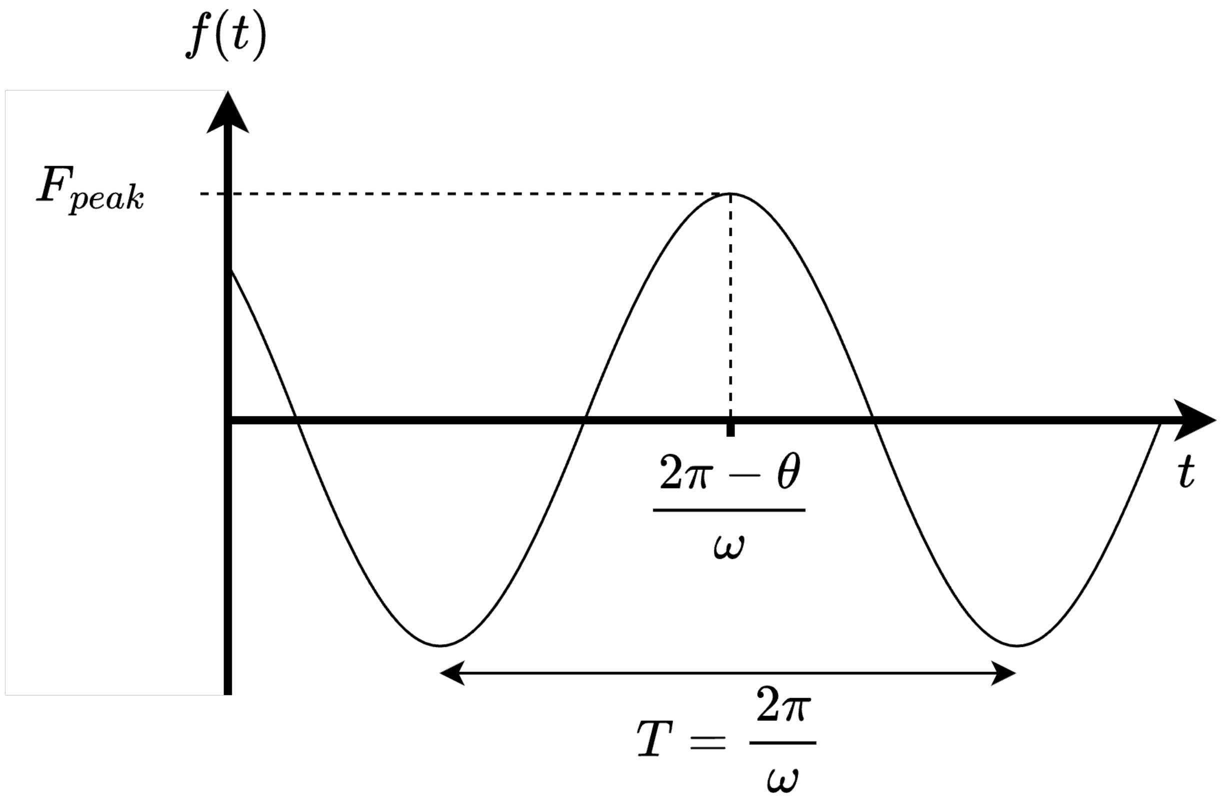
Harmonic functions

In this class, we deal with harmonic functions:

$$f(t) = F_{peak} \cos(\omega t + \theta)$$

where:

- F_{peak} is the amplitude,
- ω is the angular frequency,
- θ is the phase angle.

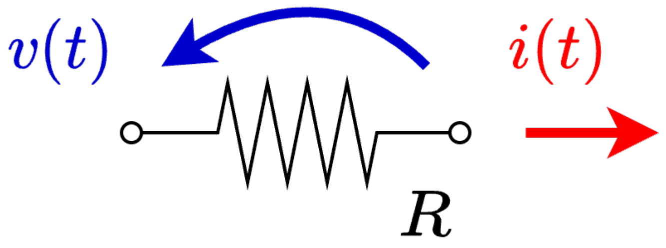


The angular frequency is linked to the frequency f and period T as

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Root Mean square (RMS) value

Let's consider a resistance R:



The instantaneous power $p(t)$ dissipated in the resistance is

$$p(t) = v(t)\, i(t) = R\, i(t)^2 = \frac{v(t)^2}{R}$$

And the mean dissipated power is

$$P = \frac{1}{T}\int_0^T v(t)\, i(t)\, dt = R\, \underbrace{\frac{1}{T}\int_0^T i(t)^2\, dt}_{I_{RMS}^2} = \frac{1}{R}\underbrace{\frac{1}{T}\int_0^T v(t)^2\, dt}_{V_{RMS}^2}$$

Root Mean square (RMS) value

The Root Mean Square (RMS) value (or effective value) of $f(t)$ is defined as:

$$F_{RMS} = \sqrt{\frac{1}{T}\int_0^T f(t)^2\,dt}.$$

With $f(t) = F_{peak}\cos(\omega t + \theta)$, we obtain:

$$F_{RMS} = \sqrt{\frac{1}{T}\int_0^T F_{peak}^2 cos^2(\omega t + \theta)\,dt} = F_{peak}\sqrt{\frac{1}{T}\int_0^T cos^2\,(\omega t)dt}$$

$$= F_{peak}\sqrt{\frac{1}{T}\int_0^T \frac{1+\cos(2\omega t)}{2}dt} = F_{peak}\sqrt{\frac{1}{T}\int_0^T \frac{1}{2}dt} = \frac{F_{peak}}{\sqrt{2}}$$

Root Mean square (RMS) value

Remark:

The following conventions will be used in the rest of the course for RMS and peak/max values:

$$F_{RMS} \quad \Rightarrow \quad F$$

$$F_{peak} \quad \Rightarrow \quad F_m$$

In this class, we deal only with harmonic functions $f(t) = F_m \cos(\omega t + \theta)$ for which we have

$$\boxed{F_m = \sqrt{2}\, F}$$

The phasors

The definition of the phasors is obtained by manipulating the expression of $f(t)$:

$$\begin{aligned} f(t) &= F_m \cos(\omega t + \theta) \\ &= \sqrt{2}\, F \cos(\omega t + \theta) \\ &= \boldsymbol{R}\left(\sqrt{2}\, F\, e^{j(\omega t + \theta)}\right) \\ &= \boldsymbol{R}\left(\sqrt{2}\, \underbrace{F\, e^{j\theta}}_{\bar{F}}\, e^{j\omega t}\right) \end{aligned}$$

$$\boxed{\bar{F} = F\, e^{j\theta}}$$

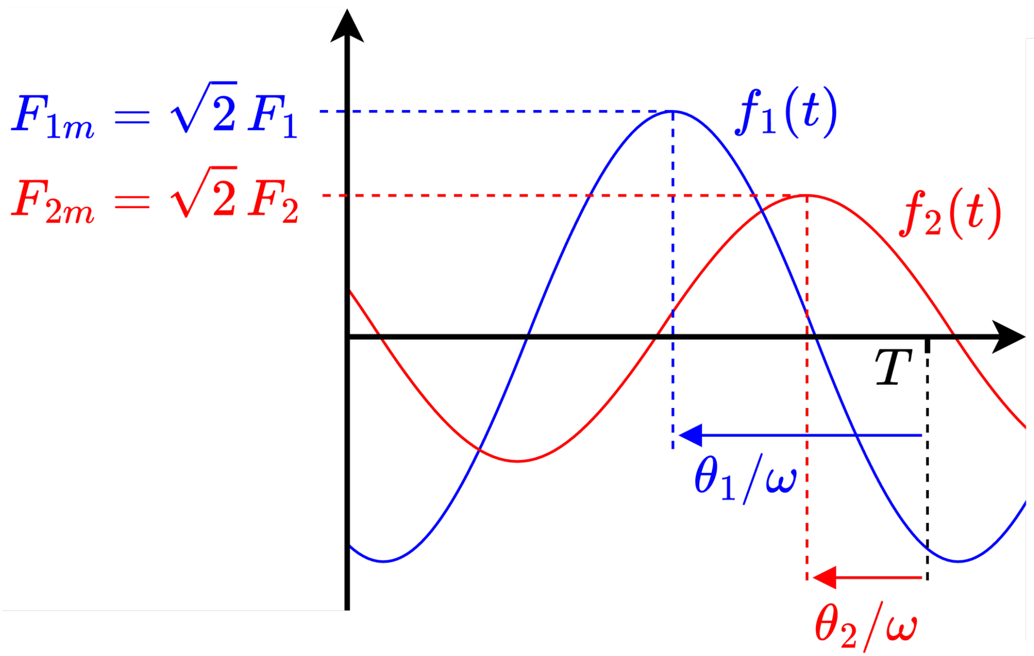
is the phasor of $f(t)$. It is a complex number with information on the signal amplitude (F) and the signal phase (θ). For a system operating at a fixed frequency, F and θ holds all the relevant information to describe the system.

The phasors

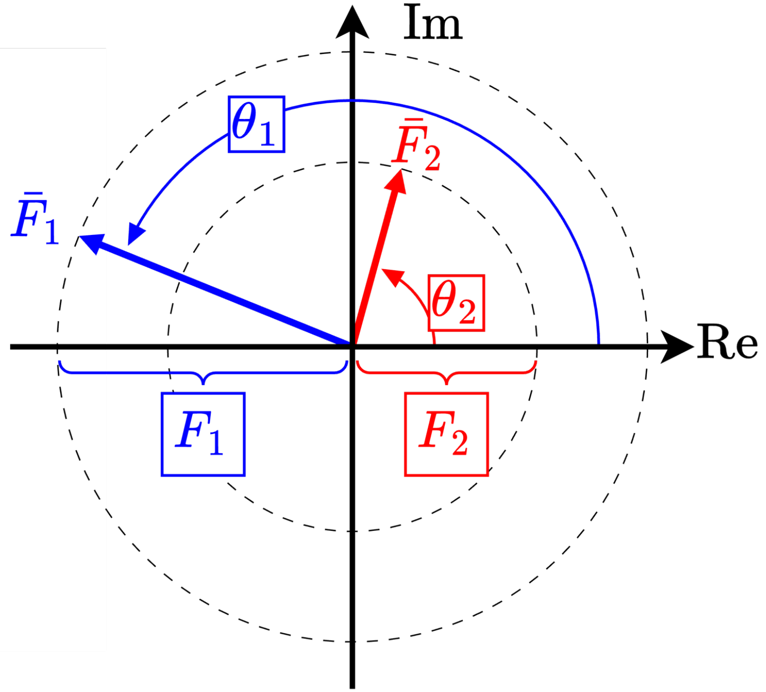
For signals **evolving at a same angular frequency ω**, phasors are like a snapshot representation containing all the necessary information (phase angle and amplitude).

$$f_1(t) = \sqrt{2}\, F_1 \cos(\omega t + \theta_1) \quad \Leftrightarrow \quad \bar{F}_1 = F_1\, e^{j\theta_1} = F_1 \angle \theta_1$$

$$f_2(t) = \sqrt{2}\, F_2 \cos(\omega t + \theta_2) \quad \Leftrightarrow \quad \bar{F}_2 = F_2\, e^{j\theta_2} = F_2 \angle \theta_2$$



$\Longleftrightarrow$



Harmonic function ⇔ Phasor equivalence

Linearity:

$$
\begin{aligned}
f_1(t) + f_2(t) &= F_{1m}\cos(\omega t + \theta) + F_{2m}\cos(\omega t + \psi) \\
&= \sqrt{2}\,F_1 \cos(\omega t + \theta) + \sqrt{2}\,F_2 \cos(\omega t + \psi) \\
&= \boldsymbol{R}\left(\sqrt{2}\,F_1\,\mathrm{e}^{j(\omega t + \theta)}\right) + \boldsymbol{R}\left(\sqrt{2}\,F_2\,e^{j(\omega t + \psi)}\right) \\
&= \boldsymbol{R}\left(\sqrt{2}\,e^{j\omega t}\,\underbrace{\left(F_1 e^{j\theta} + F_2 e^{j\psi}\right)}_{\bar{F}_1 + \bar{F}_2}\right)
\end{aligned}
$$

$$\boxed{a\,f_1(t) + b\,f_2(t) \quad \Leftrightarrow \quad a\,\bar{F}_1 + b\,\bar{F}_2}$$

Harmonic function ⇔ Phasor equivalence

Differentiation:

$$\begin{aligned}\frac{df(t)}{dt} &= \frac{d}{dt}(F_m\cos(\omega t+\theta)) \\ &= \frac{d}{dt}\,\boldsymbol{R}\left(\sqrt{2}\,F\,e^{j(\omega t+\theta)}\right) \\ &= \boldsymbol{R}\left(\sqrt{2}\,F\,e^{j(\omega t+\theta)}j\omega\right) \\ &= \boldsymbol{R}\left(\sqrt{2}\,\underbrace{j\omega\,F\,e^{j\theta}}_{j\omega\,\bar{F}}\,e^{j\omega t}\right)\end{aligned}$$

$$\boxed{\frac{df(t)}{dt} \quad\Leftrightarrow\quad j\omega\,\bar{F}}$$

Harmonic function ⇔ Phasor equivalence

Integration:

$$\int f(t)\,dt = \int F_m \cos(\omega t + \theta)\,dt$$

$$= \int \boldsymbol{R}\left(\sqrt{2}\,F\,e^{j(\omega t+\theta)}\right) dt$$

$$= \boldsymbol{R}\left(\sqrt{2}\,F\,e^{j(\omega t+\theta)}\,{}^{1}\!/_{j\omega}\right)$$

$$= \boldsymbol{R}\left(\sqrt{2}\,\underbrace{{}^{1}\!/_{j\omega}\,F\,e^{j\theta}}_{\bar{F}/_{j\omega}}\,e^{j\omega t}\right)$$

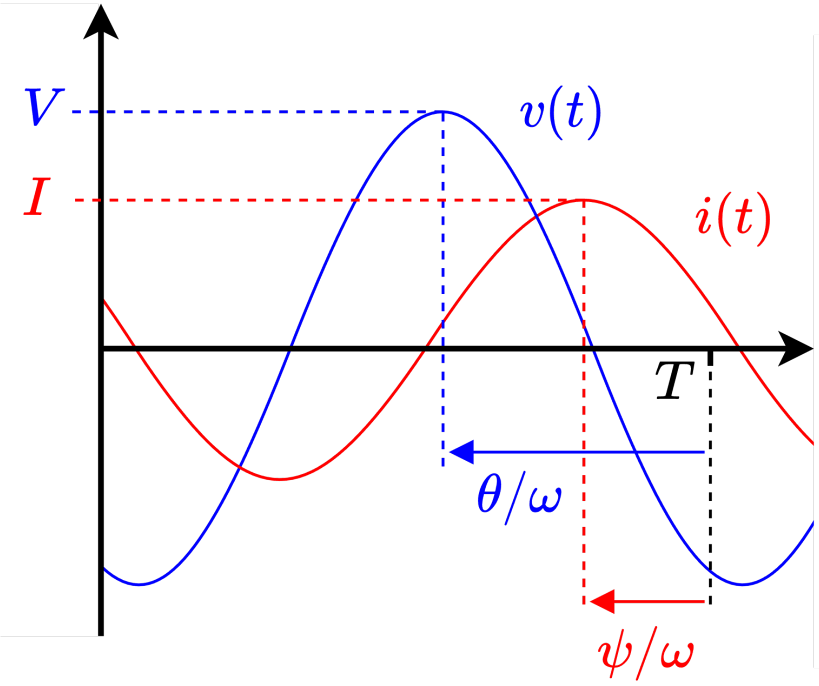
$$\boxed{\int f(t)\,dt \quad \Leftrightarrow \quad \frac{\bar{F}}{j\omega}}$$

Phasors applied to circuits

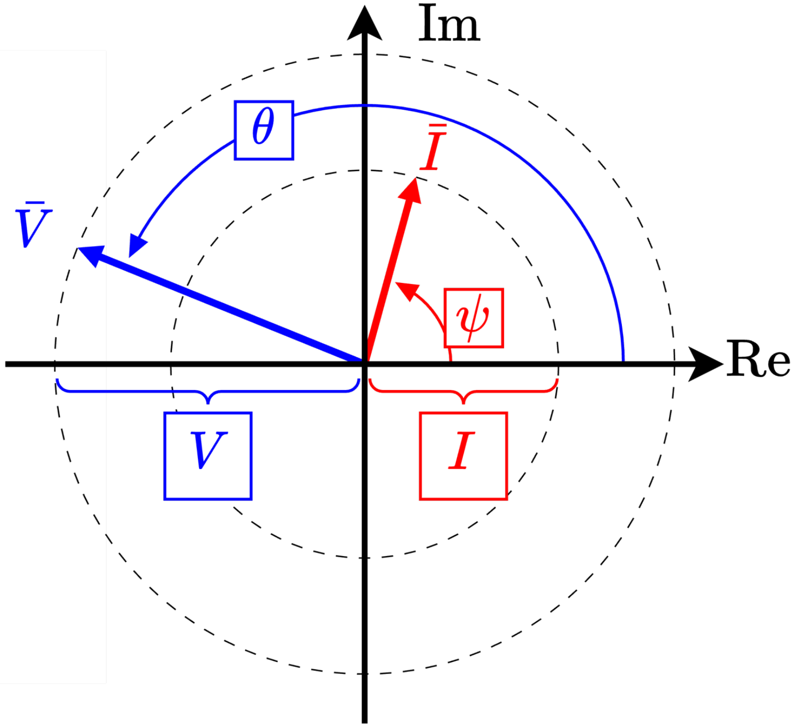
The phasor formalism can be exported to voltages and currents:

$$v(t) = \sqrt{2}\,V\cos(\omega t + \theta) \quad \Leftrightarrow \quad \bar{V} = V\,e^{j\theta}$$

$$i(t) = \sqrt{2}\,I\cos(\omega t + \psi) \quad \Leftrightarrow \quad \bar{I} = I\,e^{j\psi}$$



$\Longleftrightarrow$



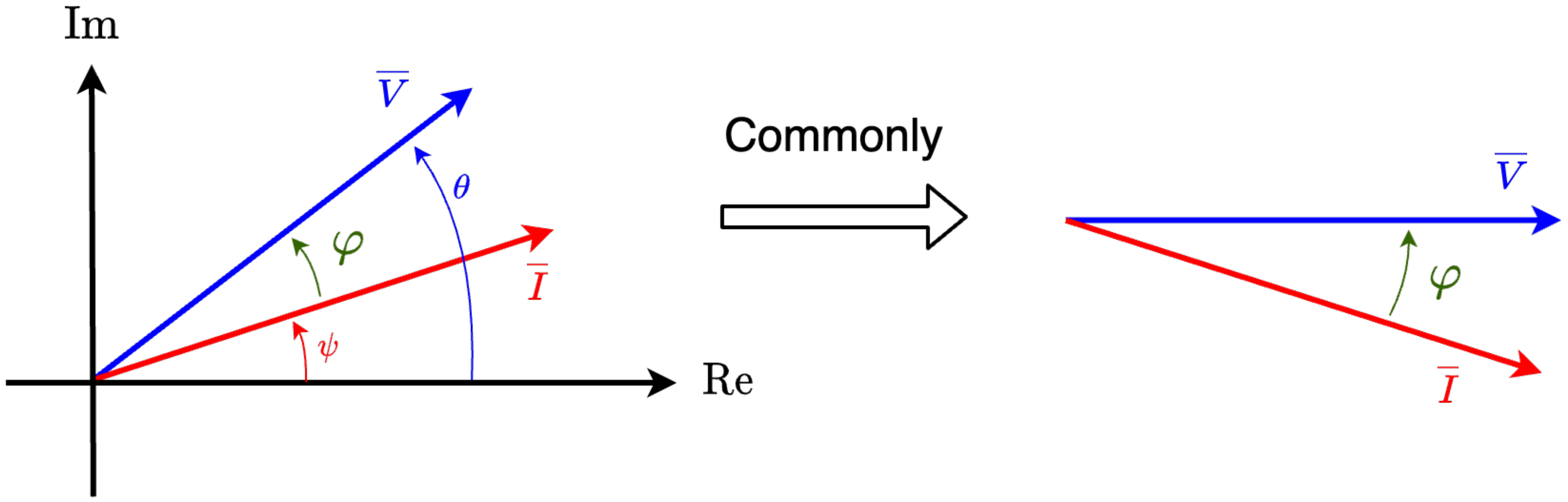
Phasors applied to circuits

The phase angle difference (or phase lag) is noted φ. It is defined as:

$$\boxed{\varphi = \theta - \psi}$$

(φ always goes from the current to the voltage !!!)

It is common to take $\bar{V}$ as the reference with $\theta = 0$ and $\varphi = -\psi$.



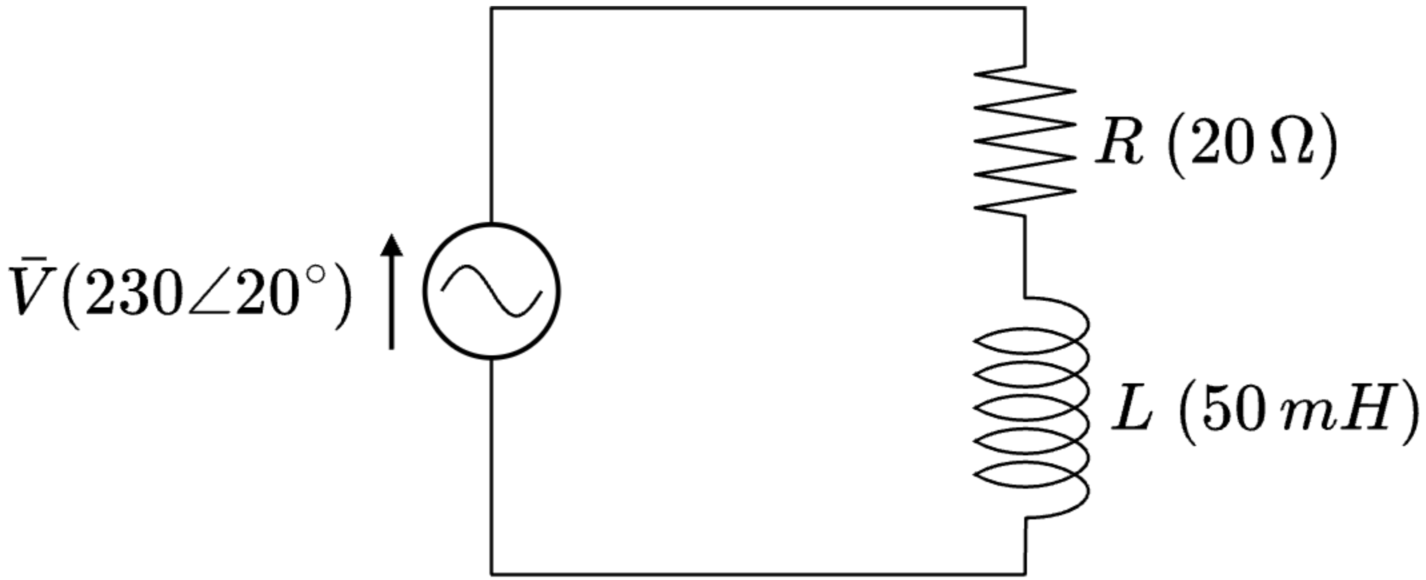
Phasors applied to circuits

	Time domain	$\Leftrightarrow$	Frequency domain
	$f(t) = \sqrt{2}\, F \cos(\omega t + \theta)$	$\Leftrightarrow$	$\bar{F} = F\, e^{j\theta}$
	$a\, f_1(t) + b\, f_2(t)$	$\Leftrightarrow$	$a\, \bar{F}_1 + b\, \bar{F}_2$
	$df(t)/dt$	$\Leftrightarrow$	$j\omega\, \bar{F}$
	$\int f(t)\, dt$	$\Leftrightarrow$	$\bar{F}/j\omega$
Resistance:	$v(t) = R\, i(t)$	$\Leftrightarrow$	$\bar{V} = R\, \bar{I}$
Inductance:	$v(t) = L\ di(t)/dt$	$\Leftrightarrow$	$\bar{V} = j\omega L\, \bar{I}$
Capacitance:	$v(t) = (1/C) \int i(t)\, dt$	$\Leftrightarrow$	$\bar{V} = \bar{I}/j\omega C$

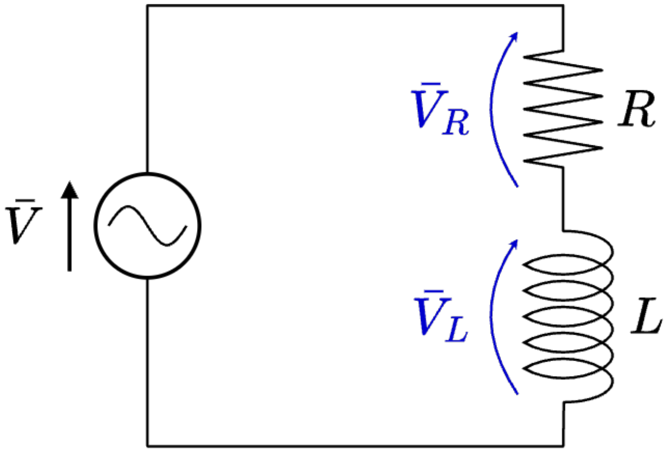
Thanks to the linearity equivalence, the Kirchhoff laws can be applied to phasors. In addition, resistances, inductances and capacitances can be replaced by complex impedances $Z = \frac{\bar{V}}{\bar{I}}$ of value respectively equal to R, $j\omega L$ and $\frac{1}{j\omega C}$.

Application: Exercise 1

For the hereunder circuit powered with a 50 Hz sinusoidal voltage generator, find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



Application: Exercise 1: Watch out !



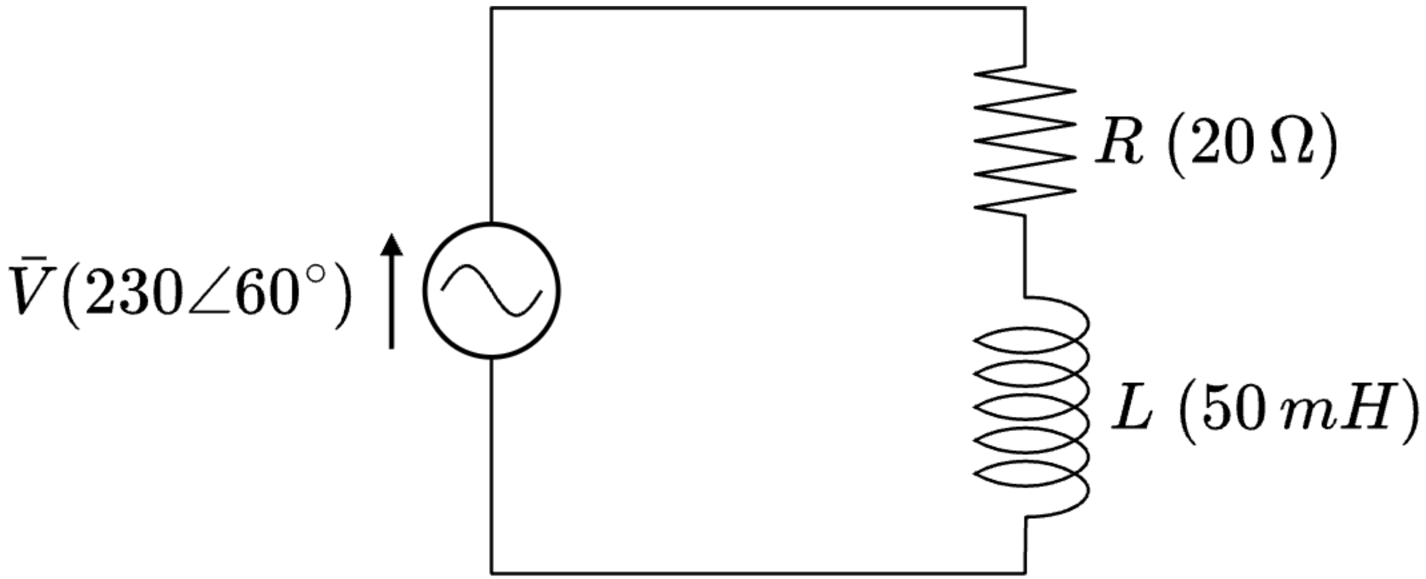
What the student writes on the exam copy:	The teaching assistant face when correcting:	Student score at the question:
$\bar{V} = \bar{V}_R + \bar{V}_L$	🤩	≥ 0
$\cancel{V = V_R + V_L}$	😢	$= 0$

$\bar{V}$, $\bar{V}_R$ and $\bar{V}_L$ are complex numbers,
V, V_R and V_L are their respective magnitudes/norms.

The sum of the norms of individual complex numbers usually differs from the norm of the sum of those complex numbers.

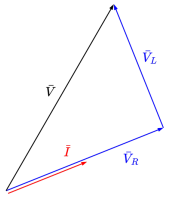
Homework 1

For the hereunder circuit powered with a 50 Hz sinusoidal voltage generator, find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



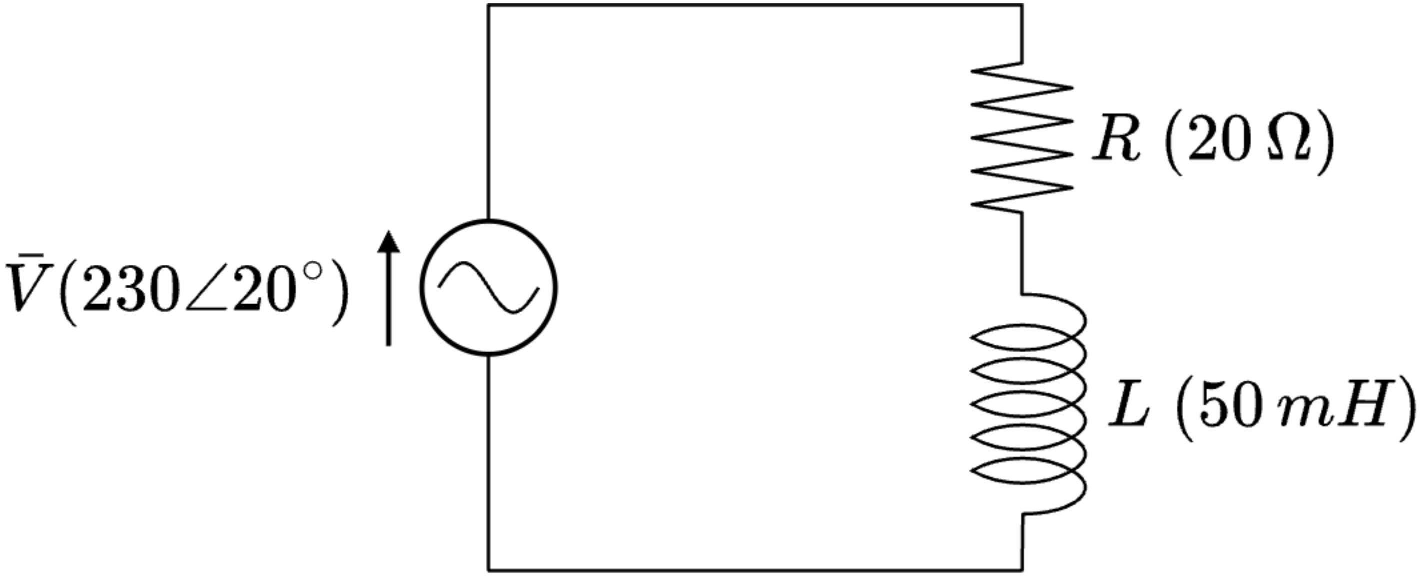
Answer:

$I = 9.044 \angle 21.854°$
$V_R = 180.88 \angle 21.854°$
$V_L = 142.06 \angle 111.854°$
(rotation of 40° compared to Ex. 1.)

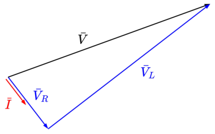


Homework 2

For the hereunder circuit powered with a 200 Hz sinusoidal voltage generator, find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



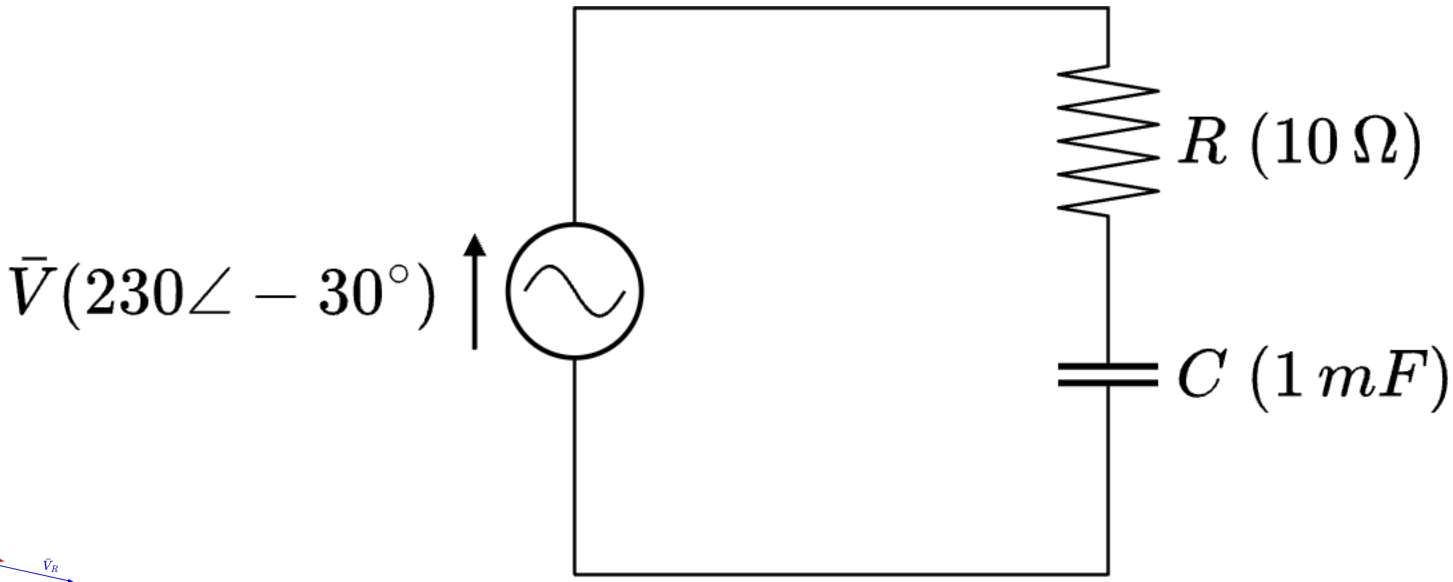
Answer:



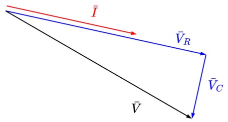
$I = 3.488 \angle -52.34°$
$V_R = 69.762 \angle -52.34°$
$V_L = 219.165 \angle 37.66°$
(Try to understand the impact of the frequency by comparison to Ex. 1.)

Homework 3

For the hereunder circuit powered with a 50 Hz sinusoidal voltage generator, find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



Answer:

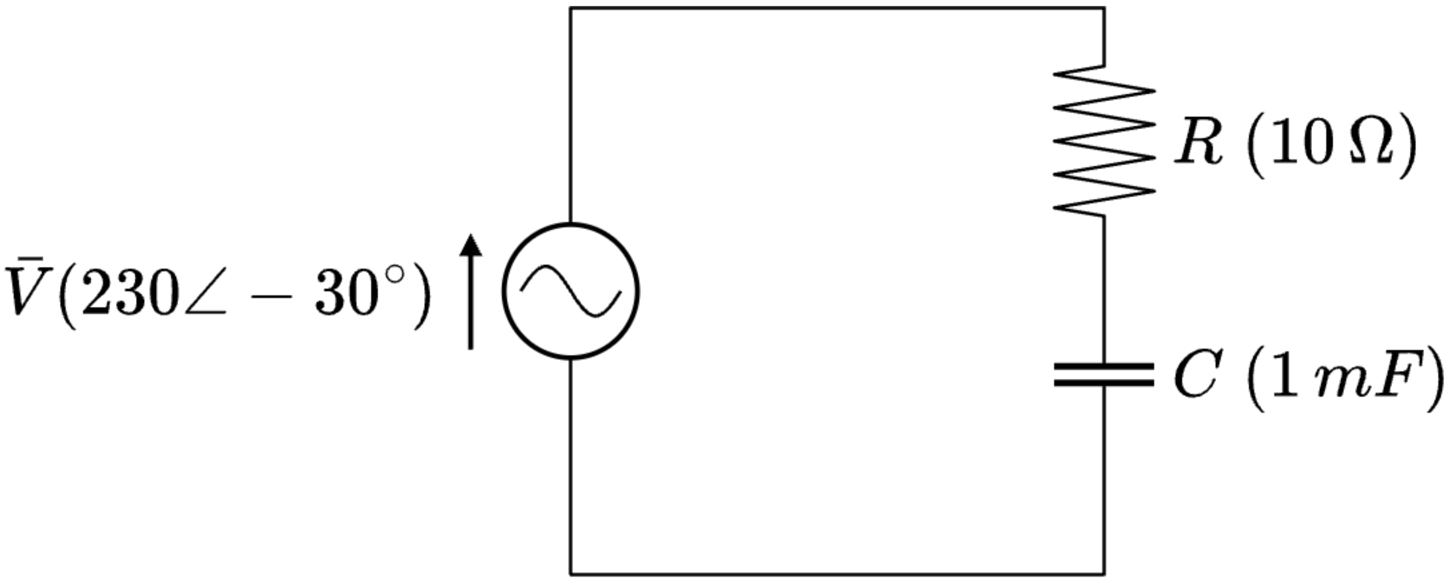


$I = 21.916\angle-12.34°$
$V_R = 219.16\angle-12.34°$
$V_C = 69.762\angle-102.34°$

(Compare the phasor diagram with the one of Ex. 1, Hw. 1 and Hw. 2:
- for a capacitor, the voltage is always delayed by 90° compared to the current through it;
- for an inductor, the voltage is always in advance by 90° compared to the current through it.)

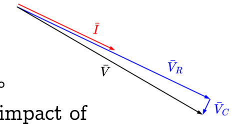
Homework 4

For the hereunder circuit powered with a 200 Hz sinusoidal voltage generator, find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



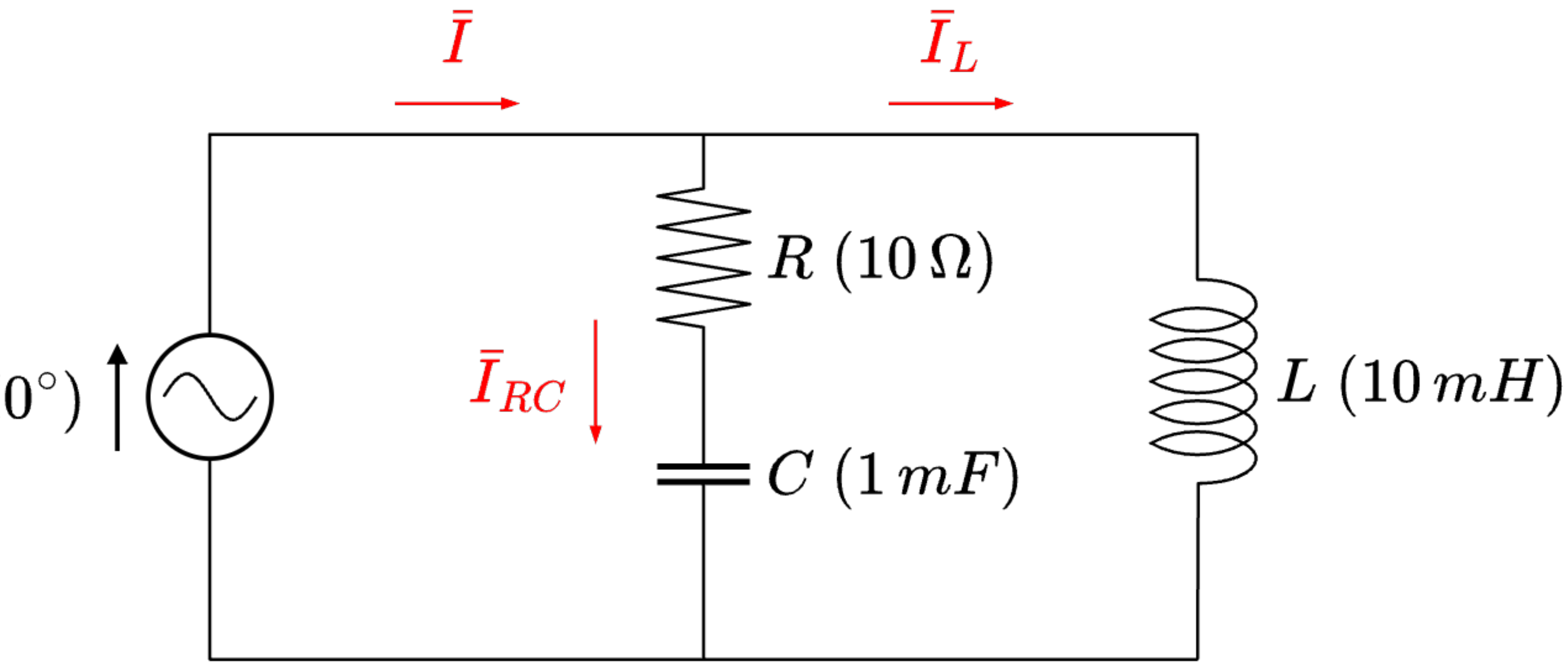
Answer:

$I = 22.928\angle -25.45°$
$V_R = 229.28\angle -25.45°$
$V_C = 18.245\angle -115.45°$
(Try to understand the impact of the frequency by comparison to Hw. 3.)

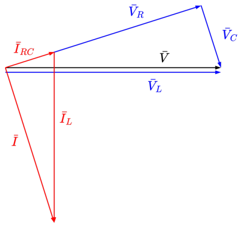


Homework 5

For the hereunder circuit powered with a 50 Hz sinusoidal voltage generator, find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



Answer:



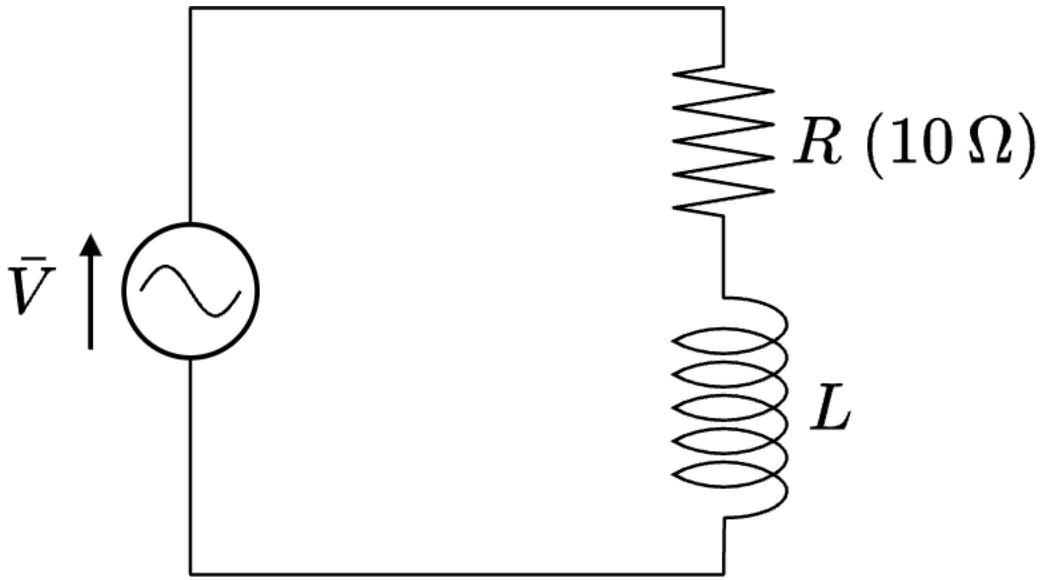
$I_{RC} = 21.915\ \angle 17.657°$
$I_L = 73.211\ \angle -90°$
$I = 69.763\ \angle -72.581°$
$V_R = 219.15\ \angle\ 17.657°$
$V_C = 69.762\ \angle -72.343°$
$V_L = 230\ \angle\ 0°$
(Note that $I = I_L + I_{RC}$ but $I \neq I_L + I_{RC}$ ➔ see slide 24)

*Answers updated on the 20/02/2024

Homework 6

A resistor R = 10 Ω and an inductor L are connected in series to a 50 Hz sinusoidal voltage generator of RMS voltage $V = 230\,V$. An RMS voltage of 100 V is measured on the terminals of the resistor. What is the value of L?

Hint:
Use the norm of $R + j\omega L$.

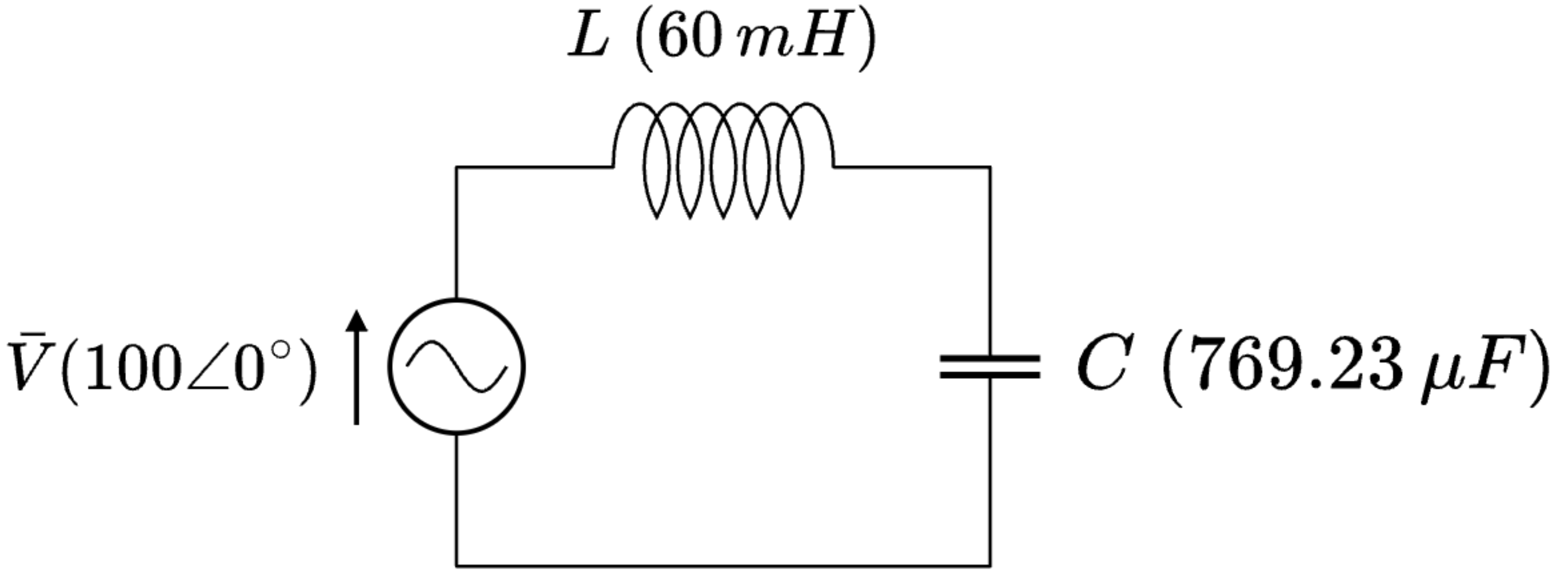


Answer:

$L = 65.929\ mH$

Homework 7

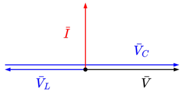
Consider the circuit hereunder operating with a pulsation $\omega = 2\pi f = 100\, rad/s$. Find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram. Do the same in case the inductance is increased to $200\, mH$.



Answer:

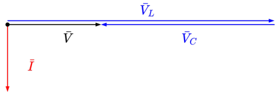
<u>$L = 60\, mH$:</u>

$I = 14.286 \angle 90^\circ$
$V_C = 185.714 \angle 0^\circ$
$V_L = 85.714 \angle 180^\circ$



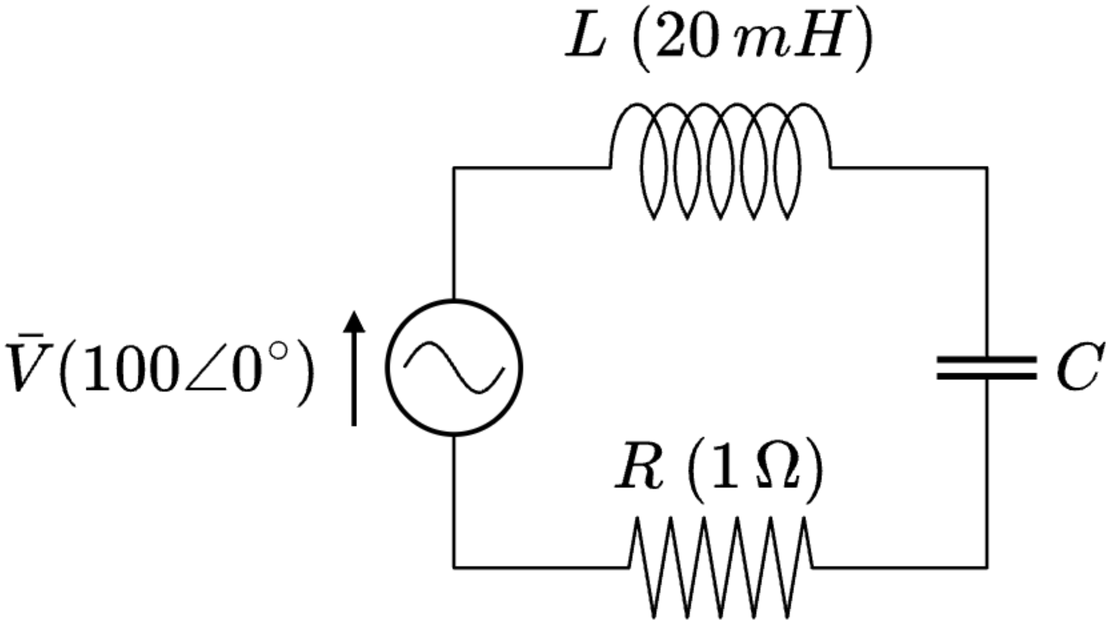
<u>$L = 200\, mH$:</u>

$I = 14.286 \angle -90^\circ$
$V_C = 185.714 \angle 180^\circ$
$V_L = 285.715 \angle 0^\circ$



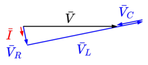
Homework 8

Consider the circuit hereunder operating at 50 Hz. What is the value of C if the RMS current is measured to be 20 A? Find the voltages and currents across all components (magnitude and phase angle). Represent all of them on a phasor diagram.



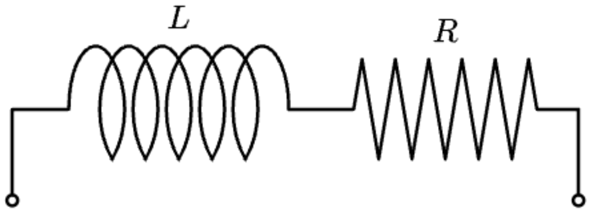
Answer:

$C = 2.3\ mF$
$I = 20 \angle -78.464°$
$V_R = 20 \angle -78.464°$
$V_C = 27.678 \angle -168.464°$
$V_L = 125.658 \angle 11.536°$



Homework 9

Consider the load showed on the right.



To determine the value of the resistance and the value of the inductance, two tests are performed:

- A DC voltage of 150 V is applied between the load terminals for a measured current of 1.95 A.
- An AC voltage of RMS value 230 V, oscillating at 50 Hz, is applied between the load terminals for a measured RMS current of 2.81 A.

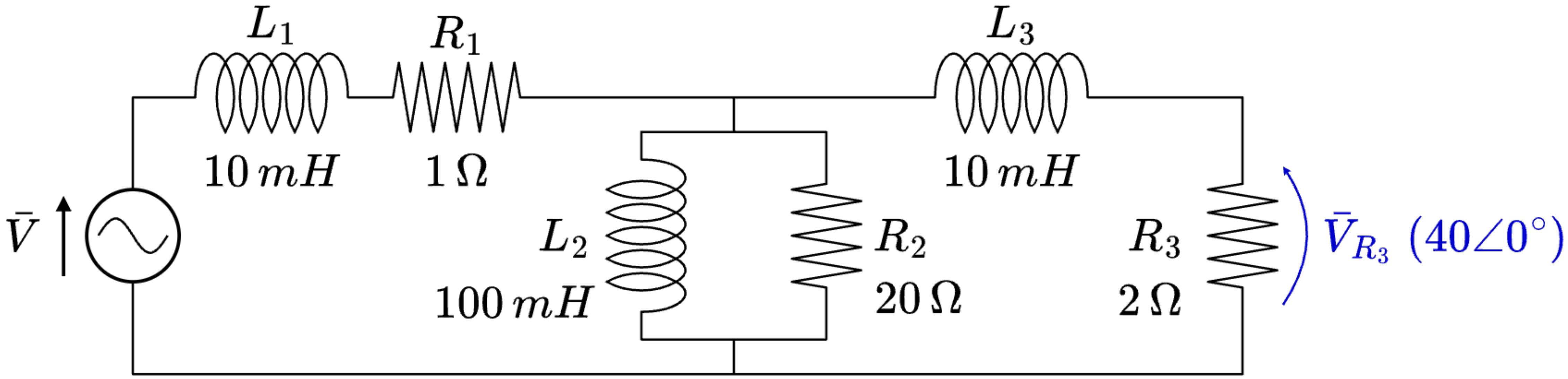
What are the values of R and L?

Answer:

$R = 76.923\ \Omega$
$L = 89\ mH$

Homework 10

Considering the circuit depicted hereunder, find the input voltage $\bar{V}$ (magnitude and phase angle).



Answer:
$V_{R_2} = 74.484 \angle 57.518°$
$I_{R_1} = 24.073 \angle 4.45°$
$V = 151.68 \angle 67.465°$

Active, reactive, complex and apparent power

Product of two harmonic functions

Active, reactive, complex and apparent power

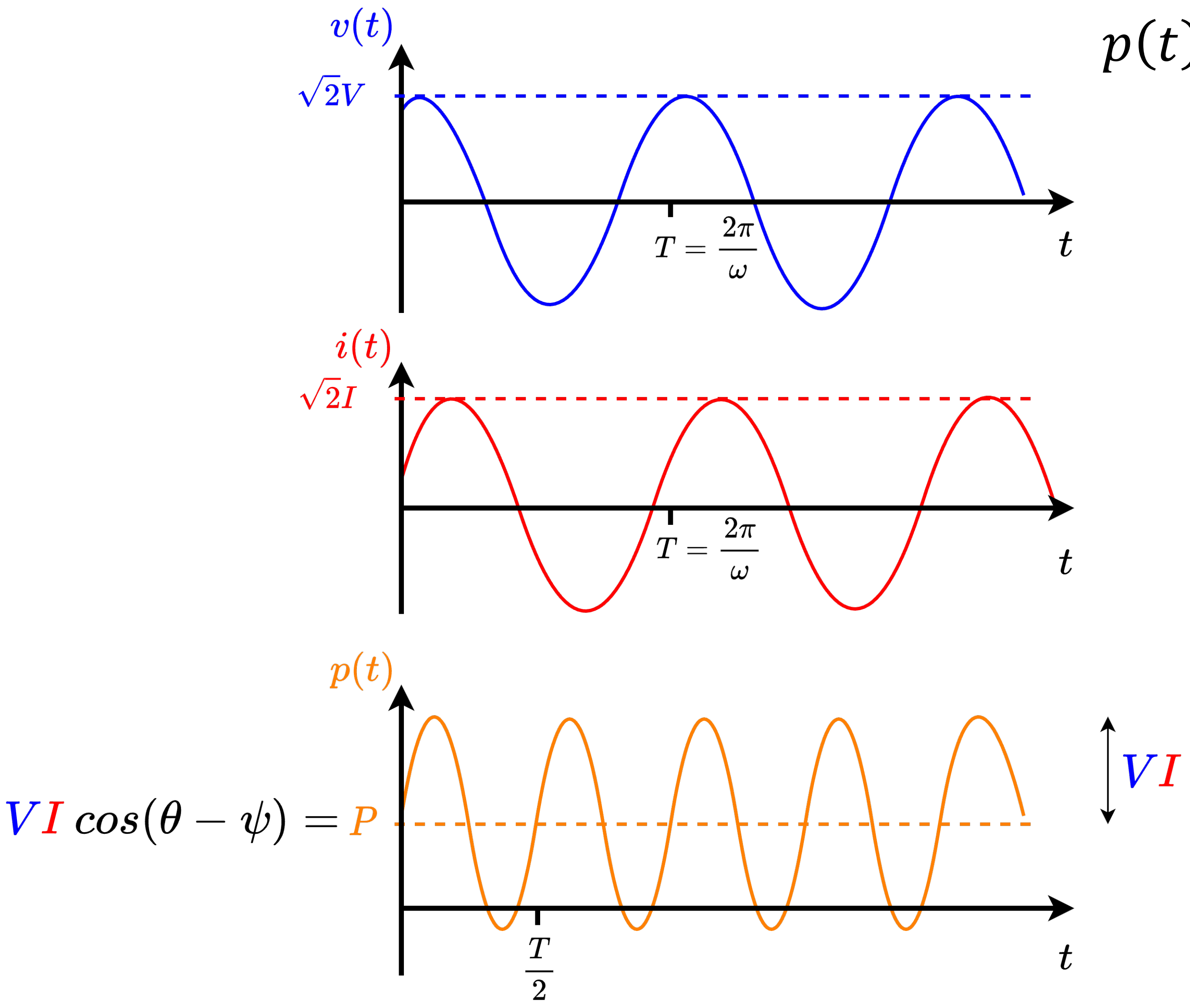
Application: Exercise 2

Product of two harmonic functions

The instantaneous power $p(t)$ consumed by passive components (or delivered by active components) is the product of the instantaneous voltage $v(t)$ with the instantaneous current $i(t)$:

$$\begin{aligned} p(t) &= v(t)\, i(t) \\ &= \sqrt{2}\, V \cos(\omega t + \theta)\ \sqrt{2}\, I \cos(\omega t + \psi) \\ &= 2\, V\, I \cos(\omega t + \theta)\ \cos(\omega t + \psi) \\ &= V\, I\, [\underbrace{\cos(\theta - \psi)}_{\text{DC component}} + \underbrace{\cos(2\omega t + \theta + \psi)}_{\text{AC component (at twice the original frequency)}}] \end{aligned}$$

Product of two harmonic functions



$$p(t) = V\,I\,[\cos(\theta - \psi) + \cos(2\omega t + \theta + \psi)]$$

The instantaneous power can be split in two parts:

- $VI\cos(\theta - \psi)$: A DC component called the Active power P.
- $V\,I\cos(2\omega t + \theta + \psi)$: An AC component oscillating at twice the original frequency.

Active, reactive, complex and apparent power

$$\begin{aligned}
p(t) &= v(t)\, i(t) \\
&= V\, I\, [\cos(\theta - \psi) + \cos(2\omega t + \theta + \psi)] \\
&= V\, I\, \big[\cos(\theta - \psi) + \cos\big(2(\omega t + \theta) - (\theta - \psi)\big)\big] \\
&= V\, I\, \big[\cos(\theta - \psi) + \cos\big(2(\omega t + \theta)\big)\cos(\theta - \psi) + \sin\big(2(\omega t + \theta)\big)\sin(\theta - \psi)\big] \\
&= \underbrace{V\, I \cos(\theta - \psi)}_{P}\, \underbrace{\big[1 + \cos\big(2(\omega t + \theta)\big)\big]}_{\geq 0} + \underbrace{V\, I \sin(\theta - \psi)}_{Q}\, \underbrace{\sin\big(2(\omega t + \theta)\big)}_{\text{Oscillating +-}}
\end{aligned}$$

$p(t)$ can be decomposed into:

- A flow of energy going always in the same direction with an average value $P = VI\ cos(\theta - \psi)$. The value P is called the active power.
- A fluctuating flow of energy, exchanged back and forth twice a period, with an average value $Q = VI \sin(\theta - \psi)$. The value Q is called the reactive power.

Active, reactive, complex and apparent power

Although $p(t) = P\left[1 + \cos\left(2(\omega t + \theta)\right)\right] + Q\ \sin\left(2(\omega t + \theta)\right)$ is a real number, one defines the complex power S as:

$$\boxed{S = P + jQ}$$

S IS NOT A PHASOR, it is just a convenient complex number allowing to describe easily and completely the instantaneous power (real number).

One also shows that is is really easy to determine S using phasors:

$$S = P + jQ = VI\cos(\theta - \psi) + j\,VI\sin(\theta - \psi) = VI\,[\cos(\theta - \psi) + j\sin(\theta - \psi)]$$
$$= VI\,e^{j(\theta-\psi)} = V\,e^{j\theta}\,I\,e^{-j\psi} = \bar{V}\,\bar{I}^*$$

$$\boxed{S = \bar{V}\,\bar{I}^*}$$

Active, reactive, complex and apparent power

One also defines the apparent power:

$$
\begin{aligned}
|S| &= \sqrt{P^2 + Q^2} \\
&= \sqrt{V^2 I^2 cos^2(\theta - \psi) + V^2 I^2 \sin^2(\theta - \psi)} \\
&= V I
\end{aligned}
$$

And the power factor:

$$PF = \cos(\theta - \psi) = \cos(\varphi) = \frac{P}{|S|}$$

Let's now suppose that we want to deliver an active power P to a load. What current I should we inject for a fixed voltage V ?

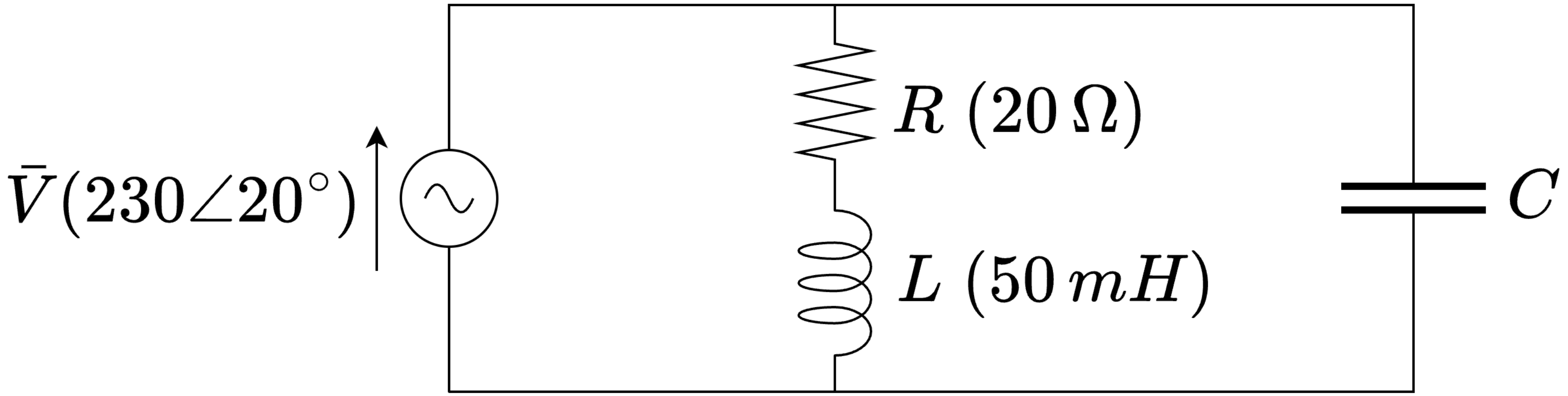
$$I = \frac{|S|}{V} = \frac{P}{V\cos(\varphi)}$$

I is larger when $\cos(\varphi)$ is small ➔ If $\cos(\varphi)$ is small, there are more transmission losses.

➔ We want $\cos(\varphi)$ to be 1 ➔ To avoid transmission losses, we want the reactive power Q to be as small as possible.

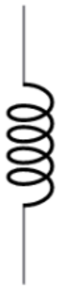
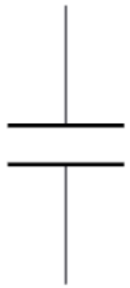
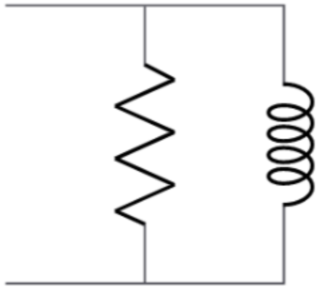
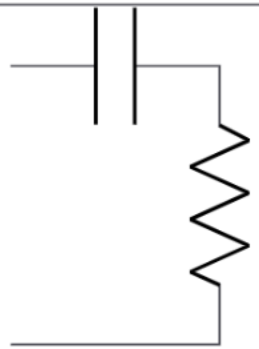
Application: Exercise 2

Your colleague suggests to add a 50 μF capacitor in parallel of the RL load. It is supposed to compensate the reactive power consumed by the inductive load. Is it a good idea ? If so, what would be the exact value of C needed ?

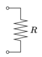
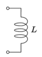
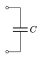
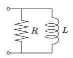
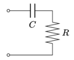


Fill the cells of the table below with the most appropriate answer among

$$= 0, \quad < 0, \quad > 0, \quad = 1, \quad < 1, \quad +\infty \text{ and } -\infty.$$

one-port:					
active power consumed					
reactive power produced					
cos ϕ					
tan ϕ					

Answer:

					
Active power consumed	> 0	$= 0$	$= 0$	> 0	> 0
Reactive power produced	$= 0$	< 0	> 0	< 0	> 0
$\cos(\varphi)$	$= 1$	$= 0$	$= 0$	> 0	> 0
$\tan(\varphi)$	$= 0$	$+\infty$	$-\infty$	> 0	< 0